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## BACKWATER FUNCTIONS BY NUMERICAL INTEGRATION

by Clint J. Keifer, A.M.ASCE and
Henry Hsien Chu

HYDRAULICS DIVISION

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# BACKWATER FUNCTIONS BY NUMERICAL INTEGRATION

By Clint J. Keifer,[1] A. M. ASCE, and Henry Hsien Chu[2]

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## Synopsis

Because of the importance of nonuniform flow and the particular difficulties in computing backwater curves in closed conduits, the authors have devised new procedures facilitating backwater computations. Complete derivations of equations for computing backwater functions are presented, and adaptations of the step method are suggested as additional aids in this field.

## Introduction

*Notation.*—The letter symbols introduced in this paper are defined where they first appear, in the text or by illustration, and are assembled alphabetically in the Appendix for convenience of reference.

Hydraulics engineers are repeatedly required to compute backwater curves. In a natural water course, the shape of the cross section may vary greatly among sections. This necessitates use of one of the many step methods. If the channel is uniform, the computations may be minimized through the integration method appropriate for the particular problem. For channels that are wide in comparison to the depth of flow, for example, one may choose the Bresse backwater functions. The method most applicable to various shapes of open channel was devised by Boris A. Bakhmeteff,[3] Hon. M. ASCE, and supplemented to include velocity head by M. E. Von Seggern,[4] A.M. ASCE.

[3] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932.

[4] "Integrating the Equation of Nonuniform Flow," by M. E. Von Seggern, *Transactions*, ASCE, Vol. 115, 1950, p. 71.

These methods cannot be applied accurately to closed conduits when flow is near the top. The writers propose a method for separating many of the involved factors in the nonuniform flow equation so that it can be integrated numerically with respect to the relative depth of flow, $y/D$, in which $y$ represents the depth of flow and $D$ is the diameter of the conduit. Circular sections have been taken as examples, and tables corresponding to the Bresse function for shallow rectangular sections have been prepared. The application of this circular section is limited, but the same principle and procedure can be applied to any type of conduit or channel.

## Numerical Integration

*Equation for Nonuniform Steady Flow.*—The equation for nonuniform steady flow can be written in the following form:

$$S = \frac{dy}{dx} + \frac{d}{dx}\left(\frac{V^2}{2\,g}\right) + S_0 \qquad (1)$$

in which $S$ is the energy gradient, $x$ represents distance (measured upstream), $V$ is the average velocity at a section, $g$ is the acceleration produced by gravity, and $S_0$ denotes the slope of the invert or bed. This equation indicates that the slope of the energy gradient is equal to the change in the depth of flow plus the velocity head, with respect to a unit length ($dx$) plus the bottom slope.

Substituting $Q^2/A^2$ for $V^2$ ($Q$ representing discharge and $A$ being the cross-sectional area of the flow) $S = \frac{dy}{dx} + \frac{d}{dx}\left(\frac{Q^2}{2\,g\,A^2}\right) + S_0$. By differentiation,

$$S = \frac{dy}{dx} - \frac{Q^2}{g\,A^3}\frac{dA}{dy}\frac{dy}{dx} + S_0 \qquad (2)$$

Since the top width of water surface $b_w = \frac{dA}{dy}$,

$$dx = \left(\frac{1 - \frac{Q^2\,b_w}{g\,A^3}}{S - S_0}\right) dy \qquad (3)$$

Eq. 3 is the general differential equation, taking $S_0$ positive when it is inclined downstream and $x$ is measured upstream. Because the expression between the parentheses in Eq. 3 is a function of depth, this equation may be written:

$$dx = f(y)\,dy \qquad (4)$$

In this form it may be integrated graphically if the size and shape of the conduit or channel, the invert slope, and the discharge are held constant. Each time one of the parameters is changed, the graphical integration must be performed again.

Eq. 3 has not been completely evaluated mathematically without approximations. Mr. Bakhmeteff has integrated Eq. 3 by assuming that the square of the conveyance at any depth of a channel varies as the $i$th power of the depth; so that, if $k$ is the conveyance and $i$ any positive number, $k^2 \propto y^i$. He also neglected the effect of the change in velocity head. This procedure necessitates the use of short steps so that the change in velocity head within each step would not be too great. Mr. Von Seggern makes the same assumptions as Mr. Bakhmeteff and introduces another approximation, $A^3/b_w \propto y^m$, which takes account of the velocity-head changes, $m$ being any positive number. These assumptions enabled both writers to integrate Eq. 3 analytically and to obtain approximate solutions for most types of open channels. The closed conduit presents a different situation in which the values of $i$ and $m$, instead of being constant, vary rapidly when the water depth approaches the crown—

thus rendering the basic assumption inapplicable. Fig. 1 shows curves describing this variation. The subscript $F$ indicates a full section. The expressions for the exponents $i$ and $m$ are

$$\left.\begin{aligned} i &= 2 \cot \theta = 2 \frac{d\left(\log \frac{k}{k_F}\right)}{d\left(\log \frac{y}{D}\right)} \\ \text{and} \qquad & \\ m &= \cot \phi = \frac{d\left(\log \frac{g\, A^3}{b_w\, D^5}\right)}{d\left(\log \frac{y}{D}\right)} \end{aligned}\right\} \qquad (5)$$

in which $\theta$ and $\phi$ are slopes of the curves in Fig. 1. Values thus determined appear in Table 1. Four parameters are present in the integrand of Eq. 4. A

TABLE 1.—VARIATION OF $i$ AND $m$ IN CIRCULAR CONDUITS

| $y/D$ | 0.30 | 0.40 | 0.50 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.94 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $i$ | 3.93 | 3.68 | 3.40 | 3.12 | 3.01 | 2.80 | 2.47 | 1.93 | 1.20 | 0.85 | 0 | −0.54 |
| $m$ | 3.87 | 3.87 | 3.87 | 3.87 | 3.89 | 3.99 | 4.01 | 4.14 | 4.42 | 4.92 | 5.98 | 6.50 |

method has been devised for removing two of the parameters from under the integral sign so that for a specified conduit shape the numerical integration must be performed only over the range of one remaining parameter. The following derivation thus required no approximations.

If a particular shape of conduit is chosen—such as a circular section—and the dimensionless ratios $y/D$ and $x/D$ are used to express depth of flow and length along the channel, then Eq. 3 becomes

$$d\left(\frac{x}{D}\right) = \left(\frac{1 - \frac{Q^2\, b_w}{g\, A^3}}{S - S_0}\right) d\left(\frac{y}{D}\right) \qquad (6)$$

$$\left.\begin{aligned} &\text{The flow } Q_{cap} = k_F \sqrt{S_0} \\ &\text{and} \\ &Q = k\sqrt{S} = k_F \sqrt{S_F} = k_o \sqrt{S_0} \end{aligned}\right\} \qquad (7)$$

in which $Q_{cap}$ represents the capacity of the conduit flowing full and having the energy gradient equal to the bottom slope, the subscript zero indicates the channel bottom, and the subscript $o$ indicates normal depth.

Therefore,

$$S = S_F \left(\frac{k_F}{k}\right)^2 = S_0 \left(\frac{Q}{Q_{cap}}\right)^2 \left(\frac{k_F}{k}\right)^2 \qquad (8)$$

Observing that $(k_F/k)^2$ is a function only of $y/D$ (designated $f_1(y/D)$) and replacing $Q/Q_{cap}$ by $\alpha$ for convenience, it is concluded that

$$S = S_0\, \alpha^2 f_1\left(\frac{y}{D}\right) \qquad (9)$$

In Eq. 6, the value of $\frac{b_w}{g\,A^3}$, after multiplying by $D^5$, is also a function of the proportional depth $y/D$—written as

$$\frac{g\,A^3}{b_w\,D^5} = f_2\left(\frac{y}{D}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10a)$$

Also,

$$f_2\left(\frac{y}{D}\right) = \frac{Q^2_c}{D^5} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10b)$$

in which the subscript $c$ corresponds to conditions at critical depth. The graphs of $f_1\ (y/D)$ and $f_2\ (y/D)$ for circular conduits are shown in Fig. 2. Substituting Eqs. 9 and $10a$ in Eq. 6,

$$d\left(\frac{x}{D}\right) = \frac{1}{S_0}\left[\frac{1 - \dfrac{Q^2}{D^5 f_2\left(\frac{y}{D}\right)}}{\alpha^2 f_1\left(\frac{y}{D}\right) - 1}\right] d\left(\frac{y}{D}\right) \ldots\ldots\ldots\ldots (11a)$$

and

$$x\Big|_{y=y_1}^{y=y_2} = \frac{D}{S_0}\int_{\frac{y_1}{D}}^{\frac{y_2}{D}} \frac{d\left(\frac{y}{D}\right)}{\left[\alpha^2 f_1\left(\frac{y}{D}\right) - 1\right]}$$

$$- \frac{Q^2}{S_0\,D^4}\int_{\frac{y_1}{D}}^{\frac{y_2}{D}} \frac{d\left(\frac{y}{D}\right)}{f_2\left(\frac{y}{D}\right)\left[\alpha^2 f_1\left(\frac{y}{D}\right) - 1\right]} \ldots (11b)$$

in which $y_1$ and $y_2$ represent flow depths at two consecutive specified sections.

Eq. $11b$ can be integrated by assigning values to $\alpha$ and integrating numerical values of $\dfrac{1}{\alpha^2 f_1\left(\frac{y}{D}\right) - 1}$ and $\dfrac{1}{f_2\left(\frac{y}{D}\right)\left[\alpha^2 f_1\left(\frac{y}{D}\right) - 1\right]}$ at each increment of 0.01 in the value of $(y/D)$, using Simpson's rule. The integration can be started at any reference value of $(y/D)$, such as zero or any convenient depth. Designating members of Eq. $11b$ by

$$\Phi\left(\frac{y}{D}, \alpha\right) = \int_0^{\frac{y}{D}} \frac{d\left(\frac{y}{D}\right)}{\alpha^2 f_1\left(\frac{y}{D}\right) - 1}$$

and

$$\Psi\left(\frac{y}{D}, \alpha\right) = \int_0^{\frac{y}{D}} \frac{d\left(\frac{y}{D}\right)}{f_2\left(\frac{y}{D}\right)\left[\alpha^2 f_1\left(\frac{y}{D}\right) - 1\right]}.$$

Eq. 11*b* becomes

$$x_{1\rightarrow 2} = \frac{D}{S_0}(\Phi_2 - \Phi_1) - \frac{Q^2}{S_0 D^4}(\Psi_2 - \Psi_1) \qquad (12)$$

For a certain shape of section, the values of Φ and Ψ are functions only of the proportional depth and α. They shall be called backwater functions. These functions have been computed for circular conduits and are exhibited in both graphical and tabular form in Figs. 3 and 4 and in Tables 2 and 3.[4a]

[4a] More extensive and detailed tables of the backwater functions may be obtained free upon request, from the authors. Address requests to Messrs. Keifer and Chu, Department of Public Works, 188 West Randolph Street, Chicago 6, Ill.

TABLE 2.—BACKWATER FUNCTION Φ FOR CIRCULAR SECTIONS

| Ratio, $\frac{y}{D}$ | Values of α | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.40 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 | 1.07 | 1.20 | 1.40 | 1.60 |
| 1.00 | 0 | 0 | 0 | 0 | 0 | | 1.161 | 0.9166 | 0.4075 | 0.2543 |
| 0.98 | 0.0235 | 0.0299 | 0.0368 | 0.0495 | 0.0817 | 0.1085 | 0.760 | 0.8567 | 0.3830 | 0.2397 |
| 0.96 | 0.0468 | 0.0592 | 0.0720 | 0.0954 | 0.1517 | 0.2777 | 0 | 0.7822 | 0.3555 | 0.2238 |
| 0.94 | 0.0700 | 0.0883 | 0.1067 | 0.1403 | 0.2188 | 0.4277 | 2.872 | 0.7016 | 0.3269 | 0.2074 |
| 0.92 | 0.0932 | 0.1173 | 0.1415 | 0.1851 | 0.2857 | 0.5763 | 5.052 | 0.6203 | 0.2982 | 0.1909 |
| 0.90 | 0.1165 | 0.1465 | 0.1764 | 0.2305 | 0.3541 | 0.7345 | 2.7491 | 0.5427 | 0.2701 | 0.1748 |
| 0.88 | 0.1398 | 0.1759 | 0.2119 | 0.2769 | 0.4257 | 0.9160 | 1.4754 | 0.4714 | 0.2432 | 0.1591 |
| 0.86 | 0.1632 | 0.2057 | 0.2481 | 0.3250 | 0.5023 | 1.1459 | 1.0082 | 0.4076 | 0.2178 | 0.1441 |
| 0.84 | 0.1868 | 0.2360 | 0.2852 | 0.3754 | 0.5868 | 1.5020 | 0.7486 | 0.3515 | 0.1942 | 0.1299 |
| 0.82 | 0.2105 | 0.2668 | 0.3236 | 0.4389 | 0.6835 | 2.5439 | 0.5803 | 0.3025 | 0.1724 | 0.1166 |
| 0.80 | 0.2344 | 0.2985 | 0.3637 | 0.4868 | 0.8002 | 0.8911 | 0.4619 | 0.2600 | 0.1524 | 0.1043 |
| 0.78 | 0.2585 | 0.3311 | 0.4059 | 0.5508 | 0.9552 | 0.6046 | 0.3740 | 0.2233 | 0.1343 | 0.0928 |
| 0.76 | 0.2830 | 0.3649 | 0.4509 | 0.6237 | 1.2090 | 0.4518 | 0.3065 | 0.1916 | 0.1179 | 0.0823 |
| 0.74 | 0.3077 | 0.4002 | 0.4997 | 0.7110 | 1.5967 | 0.3521 | 0.2533 | 0.1643 | 0.1031 | 0.0727 |
| 0.72 | 0.3329 | 0.4374 | 0.5536 | 0.8239 | 0.5844 | 0.2810 | 0.2105 | 0.1407 | 0.0899 | 0.0639 |
| 0.70 | 0.3585 | 0.4770 | 0.6151 | 0.9955 | 0.4016 | 0.2275 | 0.1765 | 0.1202 | 0.0781 | 0.0560 |
| 0.68 | 0.3848 | 0.5199 | 0.6885 | 1.5361 | 0.3001 | 0.1858 | 0.1465 | 0.1026 | 0.0676 | 0.0488 |
| 0.66 | 0.4117 | 0.5674 | 0.7831 | 0.5102 | 0.2329 | 0.1527 | 0.1228 | 0.0873 | 0.0583 | 0.0424 |
| 0.64 | 0.4394 | 0.6215 | 0.9263 | 0.3311 | 0.1845 | 0.1259 | 0.1023 | 0.0741 | 0.0500 | 0.0366 |
| 0.62 | 0.4683 | 0.6860 | 1.3682 | 0.2403 | 0.1479 | 0.1039 | 0.0857 | 0.0627 | 0.0428 | 0.0315 |
| 0.60 | 0.4985 | 0.7699 | 0.4083 | 0.1825 | 0.1196 | 0.0858 | 0.0713 | 0.0529 | 0.0364 | 0.0269 |
| 0.58 | 0.5304 | 0.8986 | 0.2596 | 0.1416 | 0.0970 | 0.0708 | 0.0593 | 0.0444 | 0.0308 | 0.0229 |
| 0.56 | 0.5647 | | 0.1851 | 0.1113 | 0.0789 | 0.0583 | 0.0492 | 0.0371 | 0.0259 | 0.0193 |
| 0.54 | 0.6022 | 0.3114 | 0.1380 | 0.0882 | 0.0640 | 0.0478 | 0.0406 | 0.0309 | 0.0217 | 0.0163 |
| 0.52 | 0.6446 | 0.1971 | 0.1051 | 0.0701 | 0.0512 | 0.0391 | 0.0334 | 0.0255 | 0.0180 | 0.0136 |
| 0.50 | 0.6947 | 0.1385 | 0.0811 | 0.0557 | 0.0410 | 0.0318 | 0.0274 | 0.0210 | 0.0149 | 0.0112 |
| 0.48 | 0.7591 | 0.1014 | 0.0628 | 0.0442 | 0.0331 | 0.0257 | 0.0221 | 0.0171 | 0.0122 | 0.0092 |
| 0.46 | 0.8585 | 0.0756 | 0.0487 | 0.0347 | 0.0265 | 0.0206 | 0.0178 | 0.0138 | 0.0099 | 0.0075 |
| 0.44 | 3.3677 | 0.0569 | 0.0377 | 0.0273 | 0.0210 | 0.0164 | 0.0142 | 0.0110 | 0.0080 | 0.0061 |
| 0.42 | 0.1837 | 0.0430 | 0.0291 | 0.0212 | 0.0164 | 0.0130 | 0.0113 | 0.0088 | 0.0063 | 0.0048 |
| 0.40 | 0.1107 | 0.0324 | 0.0223 | 0.0165 | 0.0127 | 0.0101 | 0.0088 | 0.0069 | 0.0050 | 0.0038 |
| 0.35 | 0.0419 | 0.0155 | 0.0110 | 0.0082 | 0.0064 | 0.0052 | 0.0045 | 0.0035 | 0.0026 | 0.0020 |
| 0.30 | 0.0169 | 0.0068 | 0.0049 | 0.0038 | 0.0029 | 0.0024 | 0.0021 | 0.0016 | 0.0012 | 0.0009 |
| 0.25 | 0.0062 | 0.0027 | 0.0019 | 0.0015 | 0.0012 | 0.0009 | 0.0009 | 0.0006 | 0.0005 | 0.0004 |
| 0.20 | 0.0019 | 0.0008 | 0.0006 | 0.0005 | 0.0004 | 0.0003 | 0.0003 | 0.0002 | 0.0002 | 0.0001 |
| 0.15 | 0.0004 | 0.0002 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | — | — | — |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE 3.—BACKWATER FUNCTION $\Psi$ FOR CIRCULAR SECTIONS

| Ratio, $\frac{y}{D}$ | Values of $\alpha$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.40 | 0.60 | 0.70 | 0.80 | 0.90 | 1.00 | 1.07 | 1.20 | 1.40 | 1.60 |
| 1.00 | 0 | 0 | 0 | 0 | 0 | | 0.0245 | 0.1128 | 0.0680 | 0.0482 |
| 0.98 | 0.0003 | 0.0003 | 0.0004 | 0.0006 | 0.0009 | 0.0016 | 0.0186 | 0.1120 | 0.0677 | 0.0479 |
| 0.96 | 0.0008 | 0.0010 | 0.0012 | 0.0015 | 0.0024 | 0.0053 | 0 | 0.1105 | 0.0671 | 0.0477 |
| 0.94 | 0.0015 | 0.0019 | 0.0023 | 0.0028 | 0.0043 | 0.0098 | 0.0824 | 0.1079 | 0.0663 | 0.0470 |
| 0.92 | 0.0023 | 0.0029 | 0.0036 | 0.0045 | 0.0067 | 0.0150 | 0.1614 | 0.1051 | 0.0652 | 0.0466 |
| 0.90 | 0.0034 | 0.0042 | 0.0050 | 0.0063 | 0.0096 | 0.0215 | 0.2500 | 0.1028 | 0.0641 | 0.0458 |
| 0.88 | 0.0044 | 0.0055 | 0.0067 | 0.0086 | 0.0132 | 0.0303 | 0.1852 | 0.0983 | 0.0627 | 0.0452 |
| 0.86 | 0.0058 | 0.0072 | 0.0087 | 0.0112 | 0.0175 | 0.0438 | 0.1604 | 0.0950 | 0.0613 | 0.0444 |
| 0.84 | 0.0071 | 0.0090 | 0.0110 | 0.0143 | 0.0230 | 0.0667 | 0.1439 | 0.0913 | 0.0598 | 0.0435 |
| 0.82 | 0.0087 | 0.0112 | 0.0136 | 0.0180 | 0.0299 | | 0.1321 | 0.0879 | 0.0583 | 0.0426 |
| 0.80 | 0.0106 | 0.0136 | 0.0166 | 0.0225 | 0.0389 | 0.1806 | 0.1230 | 0.0847 | 0.0568 | 0.0416 |
| 0.78 | 0.0128 | 0.0164 | 0.0204 | 0.0282 | 0.0524 | 0.1546 | 0.1153 | 0.0816 | 0.0552 | 0.0405 |
| 0.76 | 0.0150 | 0.0196 | 0.0247 | 0.0349 | 0.0768 | 0.1394 | 0.1088 | 0.0784 | 0.0537 | 0.0396 |
| 0.74 | 0.0177 | 0.0236 | 0.0300 | 0.0440 | 1.8457 | 0.1288 | 0.1032 | 0.0755 | 0.0521 | 0.0384 |
| 0.72 | 0.0206 | 0.0278 | 0.0361 | 0.0571 | 0.1950 | 0.1200 | 0.0980 | 0.0727 | 0.0505 | 0.0375 |
| 0.70 | 0.0240 | 0.0333 | 0.0443 | 0.0817 | 0.1650 | 0.1130 | 0.0933 | 0.0699 | 0.0489 | 0.0363 |
| 0.68 | 0.0280 | 0.0395 | 0.0556 | 0.1747 | 0.1465 | 0.1066 | 0.0891 | 0.0673 | 0.0473 | 0.0354 |
| 0.66 | 0.0324 | 0.0475 | 0.0724 | 0.2320 | 0.1352 | 0.1011 | 0.0851 | 0.0648 | 0.0458 | 0.0343 |
| 0.64 | 0.0377 | 0.0576 | 0.0978 | 0.1882 | 0.1263 | 0.0960 | 0.0813 | 0.0622 | 0.0442 | 0.0332 |
| 0.62 | 0.0438 | 0.0721 | 0.2500 | 0.1680 | 0.1189 | 0.0916 | 0.0777 | 0.0599 | 0.0426 | 0.0322 |
| 0.60 | 0.0511 | 0.0915 | 0.2780 | 0.1545 | 0.1120 | 0.0870 | 0.0743 | 0.0575 | 0.0411 | 0.0310 |
| 0.58 | 0.0600 | 0.1290 | 0.2265 | 0.1437 | 0.1054 | 0.0830 | 0.0710 | 0.0550 | 0.0396 | 0.0299 |
| 0.56 | 0.0705 | 0.3909 | 0.2006 | 0.1343 | 0.0997 | 0.0790 | 0.0678 | 0.0529 | 0.0381 | 0.0288 |
| 0.54 | 0.0840 | 0.3080 | 0.1828 | 0.1260 | 0.0944 | 0.0753 | 0.0647 | 0.0506 | 0.0365 | 0.0278 |
| 0.52 | 0.1014 | 0.2580 | 0.1686 | 0.1186 | 0.0895 | 0.0717 | 0.0618 | 0.0485 | 0.0351 | 0.0266 |
| 0.50 | 0.1253 | 0.2293 | 0.1562 | 0.1117 | 0.0850 | 0.0682 | 0.0588 | 0.0463 | 0.0336 | 0.0256 |
| 0.48 | 0.1609 | 0.2085 | 0.1452 | 0.1054 | 0.0803 | 0.0648 | 0.0561 | 0.0441 | 0.0321 | 0.0244 |
| 0.46 | 0.2235 | 0.1912 | 0.1350 | 0.0993 | 0.0761 | 0.0616 | 0.0533 | 0.0420 | 0.0306 | 0.0233 |
| 0.44 | 1.4143 | 0.1770 | 0.1256 | 0.0935 | 0.0718 | 0.0582 | 0.0505 | 0.0399 | 0.0291 | 0.0221 |
| 0.42 | 0.5000 | 0.1643 | 0.1170 | 0.0880 | 0.0680 | 0.0552 | 0.0478 | 0.0378 | 0.0276 | 0.0210 |
| 0.40 | 0.4116 | 0.1528 | 0.1090 | 0.0826 | 0.0641 | 0.0519 | 0.0451 | 0.0357 | 0.0261 | 0.0199 |
| 0.35 | 0.3090 | 0.1271 | 0.0917 | 0.0700 | 0.0546 | 0.0444 | 0.0386 | 0.0306 | 0.0224 | 0.0171 |
| 0.30 | 0.2432 | 0.1038 | 0.0755 | 0.0580 | 0.0454 | 0.0370 | 0.0321 | 0.0255 | 0.0188 | 0.0143 |
| 0.25 | 0.1903 | 0.0825 | 0.0607 | 0.0466 | 0.0366 | 0.0297 | 0.0258 | 0.0205 | 0.0151 | 0.0114 |
| 0.20 | 0.1425 | 0.0627 | 0.0462 | 0.0356 | 0.0281 | 0.0227 | 0.0197 | 0.0157 | 0.0115 | 0.0088 |
| 0.15 | 0.1000 | 0.0442 | 0.0327 | 0.0250 | 0.0197 | 0.0158 | 0.0138 | 0.0110 | 0.0081 | 0.0062 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Referring to Eqs. 7, when flow is at normal depth,

$$\alpha^2 f_1 \left(\frac{y_o}{D}\right) = \left(\frac{Q}{Q_{cap}}\right)^2 \frac{S_0}{S_F} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Therefore, $\alpha^2 f_1\ (y_o/D) - 1 = 0$, and the integrand in Eq. 11*a* becomes infinity. Infinite length is required to make any change in normal water depth, meaning that the backwater curve approaches the normal depth asymptotically. Eq. 10*b* indicates that, when flow is at critical depth, the numerator in Eq. 11*a* vanishes. This indicates the length of backwater curve to be zero for any change in water depth.

The heavy lines in Tables 2 and 3 show the location of normal depth. The tabulated values can be interpolated for intermediate values of $\Phi$ and $\Psi$, but not across the heavy lines, because a backwater can never pass through the normal depth. The bottom, $y/D = 0$, cannot be used as a starting point of integration

in evaluating $\Phi$ and $\Psi$ for depth above the normal depth. The zero value of $\Phi$ and $\Psi$ in Tables 2 and 3 shows where the integration has been begun in each case.

Eq. 12 indicates that the backwater functions $\Phi$ and $\Psi$ will apply to all cases, regardless of the diameter and discharge values. The usage of these functions, however, must be restricted to conduits having bottom slopes in the direction of flow either steeper or milder than the critical slope. If the bottom is horizontal, $S_0 = 0, \alpha = \infty$, and Eq. 9 becomes meaningless. In the case with adverse slope, $S_0$ is negative, and

$$Q^2_{cap} = k^2_F \, S_0 \qquad (14)$$

Therefore, $\alpha^2 = (Q/Q_{cap})^2$ is also negative. The integration process must be repeated, using these expressions to determine values of $\Phi$ or $\Psi$ for the adverse slope.

In the case of an open channel, any convenient linear measure (for example, the bottom width) can be used instead of the full depth of closed conduits. The remaining expressions can be derived in the same manner as for closed conduits.

The preparing of the tables for backwater functions for one type of section is laborious—but justified if the section will be used frequently and if many backwater curves must be computed. The step method may be preferred, however, for hydraulic investigation of conduits having other cross sections that will not be used so often.

### Backwater Curves in Uniform Channels by the Step Method

The step method for computing backwater curves has certain advantages over the integration method. It provides a complete profile of the water surface, values of the velocity at each section, and many other wanted factors.

In the step method, Eq. 3 is

$$\Delta x = \frac{\Delta\left(y + \frac{V^2}{2\,g}\right)}{S_{av} - S_0} = \frac{y_2 - y_1 + \frac{V^2_2}{2\,g} - \frac{V^2_1}{2\,g}}{S_{av} - S_0} \qquad (15)$$

in which $S_{av}$ is the average value of the slope of the energy gradient at the two ends of a reach. For computing the energy slope and velocity head, many laborious steps are involved in commonly used forms of backwater computation. To save this labor, the writers have devised two new approaches which are supplementary to the integration method.

In the first of these, from Eq. 8,

$$\frac{S_F}{S} = \frac{1}{f_1\left(\frac{y}{D}\right)} \qquad (16)$$

Because $S_F$ can be computed from given values of the discharge Q, and diameter or size D, values of $S$ for any given water depth can be determined from Eq. 16.

Also for given values of discharge and diameter the velocity head can be computed from the relationship,

$$\frac{\frac{V^2}{2g}\,(\text{full})}{\frac{V^2}{2g}} = \frac{h_F}{h_v} = f_3\left(\frac{y}{D}\right) \qquad (17)$$

in which $h_v$ is the velocity head, and $h_F$ is the velocity head of a full section.

Eqs. 16 and 17 are represented graphically in Fig. 5 for circular sections, $S_F$ having been determined by the Manning formula. Table 4 facilitates the computations of slope full and velocity head flowing full, as expressed in the following equations:

$$\left.\begin{aligned} S_F &= a\left(\frac{Q}{1{,}000}\right)^2, \quad n = 0.013 \\ h_F &= b\left(\frac{Q}{1{,}000}\right)^2 \end{aligned}\right\} \qquad (18)$$

in which $a$ and $b$ are quantities evaluated in Table 4. Fig. 5 provides a scale to

TABLE 4.—Values of Factors in Eqs. 18

| Diameter, $D$, in Feet | Coefficients | | Diameter, $D$, in Feet | Coefficients | |
|---|---|---|---|---|---|
| | $a$ | $b$ | | $a$ | $b$ |
| 1.0 | 787.8 | 25,173 | 7.0 | 0.02450 | 10.48 |
| 1.25 | 239.8 | 10,311 | 7.5 | 0.01696 | 7.956 |
| 1.5 | 90.68 | 4,972 | 8.0 | 0.01202 | 6.146 |
| 1.75 | 39.84 | 2,684 | 8.5 | 0.008701 | 4.822 |
| 2.0 | 19.55 | 1,573 | 9.0 | 0.006410 | 3.837 |
| 2.25 | 10.43 | 982.2 | 9.5 | 0.004806 | 3.091 |
| 2.5 | 5.944 | 644.4 | 10.0 | 0.003656 | 2.517 |
| 3.0 | 2.248 | 310.8 | 10.5 | 0.002819 | 2.072 |
| 3.5 | 0.9881 | 167.8 | 11.0 | 0.002199 | 1.719 |
| 4.0 | 0.4846 | 98.33 | 11.5 | 0.001735 | 1.439 |
| 4.5 | 0.2585 | 61.39 | 12.0 | 0.001383 | 1.214 |
| 5.0 | 0.1474 | 40.28 | 12.5 | 0.001112 | 1.031 |
| 5.5 | 0.08867 | 27.51 | 13.0 | 0.0009023 | 0.8814 |
| 6.0 | 0.05575 | 19.42 | 13.5 | 0.0007376 | 0.7578 |
| 6.5 | 0.03636 | 14.10 | 14.0 | 0.0006079 | 0.6553 |

aid in computing the critical depth, a more complete version of which has been published elsewhere.[5] Detailed computation procedure is shown in Table 5.

[5] "Hydraulics of Steady Flow in Open Channels," by Sherman M. Woodward and Chesley J. Posey, John Wiley & Sons, Inc., New York, N. Y., 1949, p. 42.

## TABLE 5.—Backwater Computations by the Step Methods, for a Pipe of 72-In. Diameter [a]

| Method | Depth, $y$, in Feet (1) | Ratio, $\frac{y}{D}$ (2) | Ratio, $\frac{h_F}{h_c}$ (3) | $h_c$, in Feet (4) | $y + h_c$, in Feet (5) | Ratio, $\frac{S_F}{S}$ (6) | Slope, $S \times 10^5$ (7) | Average $S \times 10^5$ (8) | $(S_{av.} - S_0) 10^5$ (9) | Change in $y + h_c$ (10) | $\Delta x$ in Feet (11) | Station in Feet (12) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3.45[b] | 0.575 | 0.352 | 1.412 | 4.862 | 0.395 | 362 | | | | | |
| | 3.55 | 0.592 | 0.378 | 1.314 | 4.864 | 0.433 | 330 | 346 | 246 | 0.002 | 0.8 | 0.8 |
| | 3.65 | 0.609 | 0.405 | 1.227 | 4.877 | 0.474 | 302 | 316 | 216 | 0.013 | 6.0 | 6.8 |
| | 3.75 | 0.625 | 0.431 | 1.152 | 4.902 | 0.511 | 280 | 291 | 191 | 0.025 | 13.1 | 19.9 |
| | 3.85 | 0.642 | 0.457 | 1.088 | 4.938 | 0.553 | 259 | 269 | 169 | 0.036 | 21.3 | 41.2 |
| First Step Method, Using Fig. 5 | 4.00 | 0.667 | 0.498 | 0.997 | 4.997 | 0.618 | 231 | 245 | 145 | 0.059 | 40.7 | 81.9 |
| | 4.25 | 0.708 | 0.570 | 0.871 | 5.121 | 0.730 | 196 | 213 | 113 | 0.124 | 110 | 192 |
| | 4.50 | 0.750 | 0.642 | 0.773 | 5.273 | 0.833 | 172 | 184 | 84 | 0.152 | 181 | 373 |
| | 4.75 | 0.792 | 0.714 | 0.695 | 5.445 | 0.935 | 153 | 162 | 62 | 0.172 | 277 | 659 |
| | 5.00 | 0.833 | 0.790 | 0.628 | 5.628 | 1.025 | 140 | 147 | 47 | 0.183 | 389 | 1,039 |
| | 5.25 | 0.875 | 0.863 | 0.575 | 5.825 | 1.098 | 130 | 135 | 35 | 0.197 | 563 | 1,602 |
| | 5.50 | 0.917 | 0.923 | 0.538 | 6.038 | 1.150 | 124 | 127 | 27 | 0.213 | 790 | 2,392 |
| | 5.75 | 0.958 | 0.970 | 0.512 | 6.262 | 1.151 | 124 | 124 | 24 | 0.224 | 933 | 3,325 |
| | Full | 1.000 | 1.000 | 0.497 | 6.497 | 1.000 | 143 | 133 | 33 | 0.235 | 713 | 4,038 |
| | 3.45[b] | | | 1.40 | 4.85 | | 355 | | | | | |
| | 3.60 | | | 1.27 | 4.87 | | 320 | 337 | 237 | 0.02 | 8 | 8 |
| | 3.70 | | | 1.20 | 4.90 | | 280 | 300 | 200 | 0.03 | 15 | 23 |
| | 3.80 | | | 1.13 | 4.93 | | 270 | 275 | 175 | 0.03 | 17 | 40 |
| | 3.90 | | | 1.06 | 4.96 | | 240 | 255 | 155 | 0.03 | 19 | 59 |
| Second Step Method, Using Fig. 6 | 4.00 | | | 1.00 | 5.00 | | 230 | 235 | 135 | 0.04 | 30 | 89 |
| | 4.25 | | | 0.87 | 5.12 | | 195 | 213 | 113 | 0.12 | 106 | 195 |
| | 4.50 | | | 0.76 | 5.26 | | 170 | 183 | 83 | 0.14 | 169 | 364 |
| | 4.75 | | | 0.68 | 5.43 | | 150 | 160 | 60 | 0.17 | 283 | 647 |
| | 5.00 | | | 0.62 | 5.62 | | 140 | 145 | 45 | 0.19 | 422 | 1,069 |
| | 5.25 | | | 0.58 | 5.83 | | 130 | 135 | 35 | 0.21 | 600 | 1,669 |
| | 5.50 | | | 0.55 | 6.05 | | 125 | 127 | 27 | 0.22 | 814 | 2,483 |
| | 5.75 | | | 0.52 | 6.27 | | 125 | 125 | 25 | 0.22 | 880 | 3,363 |
| | Full | | | 0.50 | 6.50 | | 145 | 135 | 35 | 0.23 | 658 | 4,021 |

[a] In this example, $Q$ = 160 cu ft per sec, and $S_0$ = 0.001. As computed from Eq. 18 for use in the first step method, $S_F$ = 0.00143, and $h_F$ = 0.497. [b] Critical Depth.

Another method used for obtaining the energy slope and velocity head is the preparation of charts such as Fig. 6 for circular conduits. In Fig. 6, the heavy broken lines with arrowheads show the operations when the size, discharge, and depth of flow are specified. Backwater curves can be computed by the procedure shown in the following example (Table 5).

### Illustrative Example

A circular conduit of 72-in. diameter has a discharge of 160 cu ft per sec, a bottom slope of 1 ft per 1,000 ft and a Manning friction factor of $n = 0.013$. The distance is to be determined between the control section at critical depth and the section at which the conduit becomes full. The discharge capacity $Q_{cap}$ equals $\frac{1.486}{n} A_F R^{\frac{2}{3}}{}_F S^{\frac{1}{2}}{}_0$ or 134 cu ft per sec (with $R_F$ denoting the hydraulic radius of full section). Since $Q/D^{\frac{5}{2}} = 1.82$, the critical depth is obtained from Fig. 5 as $\frac{y_c}{D} = 0.575$, or $y_c = 3.45$. The type of backwater curve is $M_2$ in which the water depth $y$ is less than $y_o$ but greater than $y_c$.

*Integration Method.*—From Eq. 12, $x_{1\to 2} = \frac{D}{S_0}(\Phi_2 - \Phi_1) - \frac{Q^2}{S_0 D^4}(\Psi_2 - \Psi_1)$. Also, $\alpha = \frac{160}{134} = 1.2$, $\frac{D}{S_0} = 6{,}000$ ft, and $\frac{Q^2}{S_0 D^4} = 19.960 \frac{\text{ft}^2}{\text{sec}^2}$. Thus, $x_{1\to 2} = 5{,}245 - 1{,}166 = 4{,}079$ ft.

*Step Methods.*—The computations using the first step method are shown in Table 5 with the aid of Fig. 5. The distance is $x_{1\to 2} = 4{,}038$ ft. The computations using the second step method appear in Table 5 with the aid of Fig. 6. This procedure yields a value of $x_{1\to 2}$ equal to 4,021 ft.

### Conclusions

1. Most natural or artificial channels contain nonuniform flow. Uniform flow, which is usually assumed in making hydraulic computations, is actually rare. New methods are presented for solving nonunifiorm flow in prismatic channels, and the almost universal presence of nonuniform flow is emphasized.

2. Tables of backwater functions $\Phi$ and $\Psi$ for circular conduits are presented in this paper. The computation of backwater curves is considerably shortened by the use of these tables, as demonstrated by the example. Tables could be prepared as needed for any other types of conduits.

3. The time-consuming step method computations of the energy gradient and the velocity head can be simplified by the two devices suggested in this paper. The writers believe these to have been heretofore unpublished. Both procedures require the preparation of charts before hand. Fig. 5 will yield more accurate results than Fig. 6 although values in Fig. 5 are expressed as a percentage of those of the entire section, and one step of conversion is needed to determine the actual values. Furthermore, more labor is required in preparing Fig. 6 than Fig. 5. Therefore, the first method seems preferable. Charts similar to those used with the first method can also be prepared for open-channel sections; any convenient linear measure may be substituted for depth full in closed conduits (for example, bottom width).

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## APPENDIX. NOTATION

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The following symbols, adopted for use in the paper, and for the guidance of discussers, conform essentially to American Standard Letter Symbols for Hydraulics (ASA—Z10.2—1942), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:

$A$ = the area of the cross section, in square feet;
$b_w$ = the top width of the water surface, in feet;
$D$ = diameter of the conduit, in feet;
$g$ = acceleration produced by gravity;
$h_F$ = the velocity head when flowing full, in feet;
$h_v$ = the velocity head, in feet;
$i,m$ = numerical exponents;
$k$ = $\frac{1.486}{n} A R^{2/3}$, the conveyance of a cross section;
    $k_F$ = the conveyance at a full section;
    $k_o$ = the conveyance at normal depth;
$n$ = Manning's coefficient of roughness;
$Q$ = discharge, in cubic feet per second;
    $Q_c$ = discharge when flow is at critical depth;
    $Q_{cap}$ = capacity of conduit flowing full, with the slope of the energy gradient equal to the bottom slope;
$R$ = the hydraulic radius;
$S$ = the slope of the energy gradient;
    $S_F$ = the slope of the energy gradient when the discharge is flowing through a full section;
    $S_{av}$ = the average value of the slope of the energy gradient at the two ends of a reach;
    $S_0$ = slope of the invert or bottom;
$V$ = the average velocity in a section, in feet per second;
$x$ = the distance, in feet, measured upstream;
$x_{1\rightarrow 2}$ = a specific reach;
$y$ = the depth of flow, in feet;
    $y_o$ = the depth of normal flow, in feet;
    $y_1, y_2$ = flow depths at specific sections;
    $y_c$ = critical depth;
$\alpha$ = the ratio of the discharge to the capacity of a conduit flowing full; and
$\Phi, \Psi$ = backwater functions.

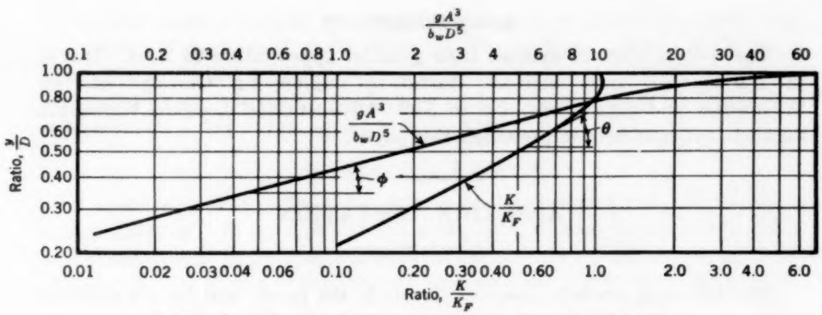


FIG. 1.—DERIVATION OF EXPONENTS $j$ AND $m$ IN CIRCULAR CONDUITS

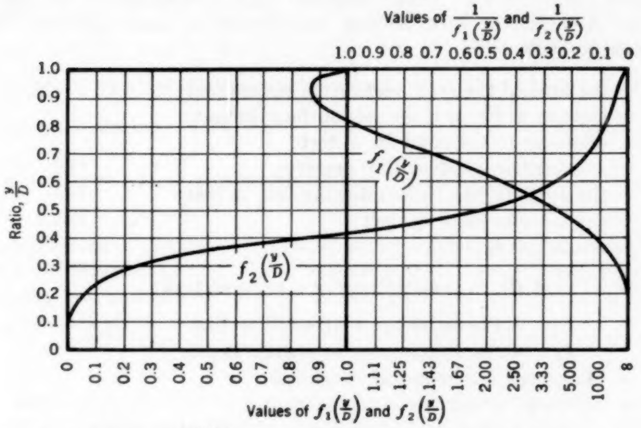


FIG. 2.—VARIATION OF $f_1\left(\frac{y}{D}\right)$ AND $f_2\left(\frac{y}{D}\right)$ WITH DEPTH, FOR CIRCULAR CONDUITS

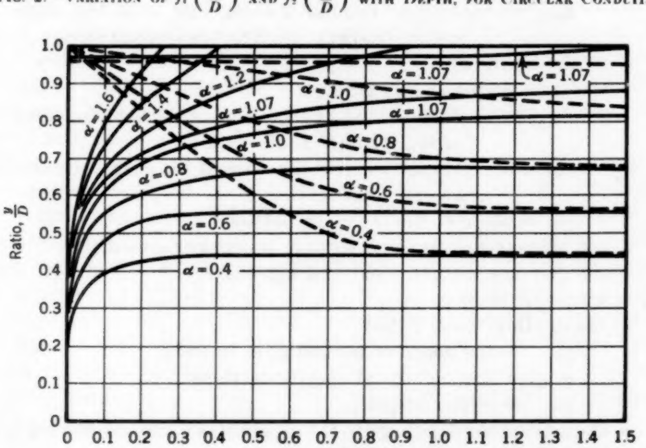


FIG. 3.—BACKWATER FUNCTION Φ, FOR CIRCULAR CONDUITS

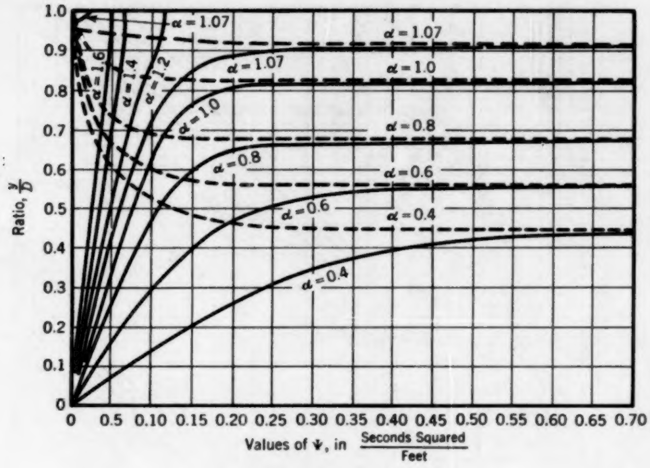


FIG. 4.—BACKWATER FUNCTION $\Psi$, FOR CIRCULAR CONDUITS

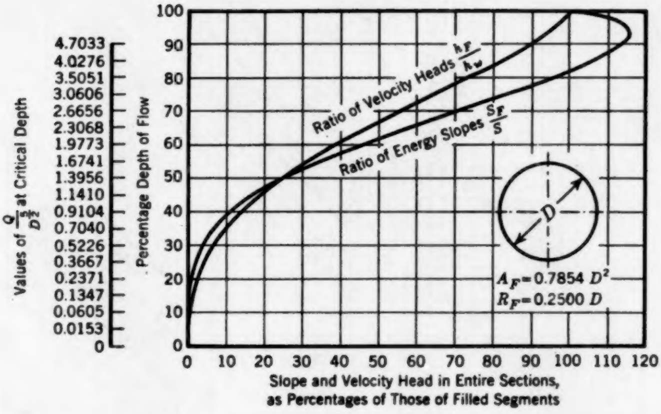


FIG. 5.—ENERGY SLOPE AND VELOCITY HEAD IN CIRCULAR CONDUITS FLOWING PARTLY FULL

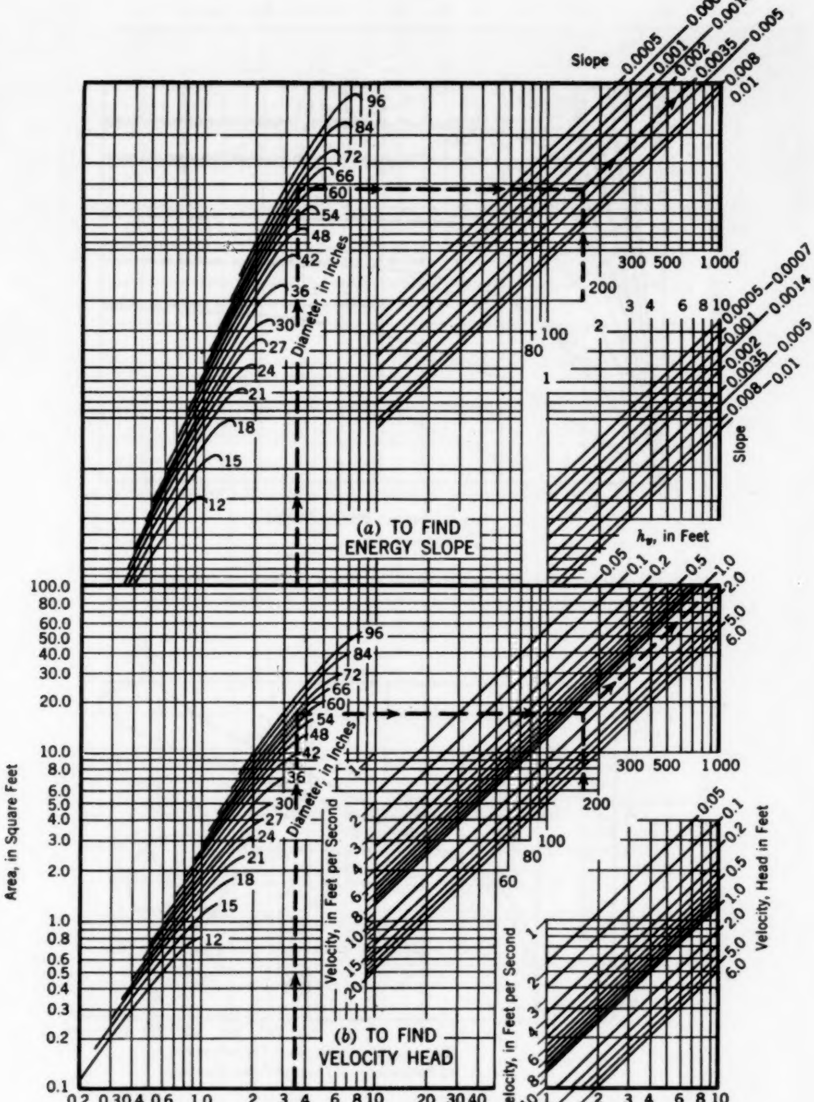


FIG. 6.—ELEMENTS OF FLOW IN CIRCULAR CONDUITS FLOWING PARTLY FULL